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4308. Proposed by Leonard Giugiuc and Sladjan Stankovik.

Let a, b and c be positive real numbers. Prove that

$$27abc(a^2b + b^2c + c^2a) \leq (a + b + c)^2(ab + bc + ca)^2.$$

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First note that $27abc(a^2b + b^2c + c^2a) \leq (a + b + c)^2(ab + bc + ca)^2 \Leftrightarrow$

$$(1) \quad 27\left(\frac{a}{c} + \frac{b}{a} + \frac{c}{b}\right) \leq (a + b + c)^2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2.$$

$$A := \frac{a}{b} + \frac{b}{c} + \frac{c}{a}, \quad B := \frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

Note that transposition a with b transform $\frac{a}{c} + \frac{b}{a} + \frac{c}{b}$ to $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and does not change $(a + b + c)^2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2$. Hence, inequality (1) holds for any positive a, b, c if it holds for any positive a, b, c such that $\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$.

Since $\frac{a}{c} + \frac{b}{a} + \frac{c}{b} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \Leftrightarrow (b - c)(a - b)(a - c) \geq 0$ for positive a, b, c then assuming $a = \max\{a, b, c\}$ (due to cyclic symmetry of (1)) suffices to prove inequality (1) for $a \geq b \geq c > 0$. In that case denoting $p := \frac{a}{b} \geq 1, q := \frac{b}{c} \geq 1$ we

obtain $b = qc, a = pqc, p, q \geq 1$ and then inequality (1) becomes

$$(2) \quad 27\left(pq + \frac{1}{p} + \frac{1}{q}\right) \leq (pq + q + 1)^2\left(\frac{1}{pq} + \frac{1}{q} + 1\right)^2, p, q \geq 1.$$

Let $u := p + q \geq 2, v := pq \geq 2$. Then we obtain

$$(pq + q + 1)^2\left(\frac{1}{pq} + \frac{1}{q} + 1\right)^2 - 27\left(pq + \frac{1}{p} + \frac{1}{q}\right) = \frac{(p + q)(2p^3q^3 + 8p^2q^2 - 19pq + 2) + (pq + 1)^2(p + q)^2 + p^4q^4 - 21p^3q^3 + 11p^2q^2 + 6pq + 1}{p^2q^2} =$$

$$\frac{u(2v^3 + 8v^2 - 19v + 2) + (v + 1)^2u^2 + v^4 - 21v^3 + 11v^2 + 6v + 1}{p^2q^2} \geq 0 \text{ (because}$$

$2v^3 + 8v^2 - 19v + 2 = (v - 1)(2v^2 + 10v - 9) - 7 \geq -7, v^4 - 21v^3 + 11v^2 + 6v + 1 \geq -2, (v + 1)^2 \geq$

and, therefore, $u(2v^3 + 8v^2 - 19v + 2) + (v + 1)^2u^2 + v^4 - 21v^3 + 11v^2 + 6v + 1 \geq$

$$4u^2 - 7u - 2 = (4u + 1)(u - 2) \geq 0).$$